Sine-Gordon-like action for the superstring in $AdS_5 \times S^5$

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ABSTRACT: We propose an action for a sine-Gordon-like theory, which reproduces the classical equations of motion of the Green-Schwarz-Metsaev-Tseytlin superstring on $AdS_5 \times S^5$. The action is relativistically invariant. It is a mass-deformed gauged WZW model for $SO(4,1) \times SO(5)/SO(4) \times SO(4)$ interacting with fermions.

KEYWORDS: AdS-CFT Correspondence, Superstrings and Heterotic Strings, Integrable Field Theories.

Contents

1. Introduction and summary

Quantizing the superstring in $AdS_5 \times S^5$ is important for understanding string theory in curved spaces and the AdS/CFT correspondence. The most successful route so far is to make use of the yet to be proven integrability of the superstring theory in $AdS_5 \times S^5$ in the light-cone gauge [1]. But the light-cone gauge-fixed worldsheet theory is a rather unusual theory from the point of view of integrable models, as it is not relativistically invariant. Although the progress in understanding the worldsheet integrability has so far defied this

point, it may nevertheless be of interest to obtain a formulation of the theory as a relativistically invariant integrable theory. In this paper we will make a step in this direction.

The idea is to find a reformulation of the model in terms of a two-dimensional, Lorentz invariant sigma-model, which is a mass deformation of a conformal field theory. This point of view has been very useful in order to construct the quantum conserved charges of integrable theories. It has been used e.g. by Reshetikhin and Smirnov [2] in the context of Sine-Gordon theory and perturbed minimal models, and by Bernard and LeClair [3] who construct the quantum non-local charges for the sine-Gordon model from the mass-deformed conformal theory of a free boson, or more generally for affine Toda theories by means of mass-deformed WZW models. Key to this approach is that the spectrum of the UV conformal theory is known. Such a formulation of the Green-Schwarz-Metsaev-Tseytlin (GSMT) string for $AdS_5 \times S^5$ [1] is missing, and we wish to propose such a reformulation. We follow the proposal of Bakas, Park and Shin (BPS) [4], which allows to construct for a bosonic symmetric space sine-Gordon model a classically equivalent theory as a mass deformed gauged WZW model.

There are various caveats with this approach, which will require further study. Firstly this reformulation is on a purely classical level. More precisely, we will construct a sigma-model, which is similar to the BPS models except that we include fermions. This sigma-model will reproduce the classical equations of motion of the GSMT superstring on $AdS_5 \times S^5$. However the Poisson structures of the two theories differ. Thus, not even classically, these are equivalent theories. But surprisingly, this does not yet imply that the quantum theories are different. A similar situation occured in [5], where two different classical Poisson structures correspond to expansion around different classical vacua of the same quantum model (see also [6]). Secondly, it would be desirable to obtain a theory that is world-sheet supersymmetric. The model that we propose may be world-sheet supersymmetric, however, we were so far unable to uncover this structure. It remains to be seen also, whether the perturbation of the underlying gauged WZW model can be computed rigorously. We leave this for the future.

The plan of this paper is as follows. We first review the boost-invariant symplectic structure of the GSMT string (Section 2). Then we review the action of Bakas, Park and Shin (sections 3.1 and 3.2) and discuss subtleties with zero modes and the relation to the Hamiltonian reduction of the WZW model (section A.1). We then propose (in section 4) the BPS-type action for the GSMT string in $AdS_5 \times S^5$ and show that it reproduces the correct equations of motion.

2. The boost-invariant symplectic structure of the Metsaev-Tseytlin superstring

2.1 Classical superstring in terms of currents

The boost-invariant symplectic structure of the classical superstring in $AdS_5 \times S^5$ was constructed in [7] in the lightcone formalism. In this formalism the classical string solution is described in terms of the data on the characteristic. The characteristic is a light-like curve on the string worldsheet. We will pick a characteristic which is described in the

conformal coordinates (τ^+, τ^-) by $\tau^- = 0$. With the appropriate choice of the boundary conditions the string phase space can be described in terms of the lightcone components of the currents at $\tau^- = 0$. The currents J take values in $\mathfrak{g} = \mathfrak{psu}(2,2|4)$, and the index $0 \ldots 3$ indicates the $\mathbb{Z}_4$ grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$:

$$J_+(\tau^+, 0) = J_{0+} + J_{1+} + J_{2+} + J_{3+} \,. \tag{2.1}$$

There are gauge transformations:

$$\delta_\xi J_+ = \partial_+ \xi + [J_+, \xi] \,, \ \xi \in \mathfrak{g}_0 \,. \tag{2.2}$$

To summarize:

$$\mathfrak{g}_0 = \mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$$
$$\mathfrak{g}_0 + \mathfrak{g}_2 = \mathfrak{so}(2,4) \oplus \mathfrak{so}(6)$$
$$\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = \mathfrak{psu}(2,2|4) \,.$$

We will introduce the notation

$$\nabla_\pm = \partial_\pm + \mathrm{ad}(J_{0\pm}) \,. \tag{2.3}$$

2.2 Geometrical meaning of $J_\pm$ and $\nabla_\pm$

Geometrically $J_{\bar{2}\pm}$ are the "lightcone velocity vectors" of the string worldsheet. In the near-flat space expansion (see [9]) they become $\partial_\pm x + (\vartheta, \Gamma \partial_\pm \vartheta) + \ldots$; both $J_{\bar{2}+}$ and $J_{\bar{2}-}$ are elements of the tangent space to $AdS_5 \times S^5$. The $\mathfrak{g}_{\bar{0}}$-components $J_{\bar{0}\pm}$ should be identified with the Levi-Civita connection (Christoffel symbols). The components in $\mathfrak{g}_{\bar{1}, \bar{3}}$ are the velocities of the worldsheet fermions, they are $J_{\bar{1}\pm} = \partial_\pm \vartheta_R + \ldots$ and $J_{\bar{3}\pm} = \partial_\pm \vartheta_L + \ldots$ in the near-flat space expansion. (In the flat space limit ϑ_L and ϑ_R would come from the left and right sectors of the worldsheet theory.)

2.3 Poisson brackets in the lightcone description

The J_+ components are independent functions of τ^+. The J_- components can be, at least formally, expressed through them using the equations of motion. Therefore the data (2.1) with the gauge equivalence (2.2) determines the string worldsheet. The string worldsheet action is degenerate, and there are additional local symmetries besides (2.2). The kappa-symmetries are partially fixed by the conditions

$$J_{1+} = J_{3-} = 0 \,. \tag{2.4}$$

We will assume (2.4) throughout this paper. It is useful to remember that with $J_{1+} = J_{3-} = 0$ the equations of motion for J_2 are

$$\nabla_\mp J_{2\pm} = 0 \,. \tag{2.5}$$

The boost-invariant lightcone Poisson brackets are

$$\{J_{0+}, J_{0+}\}^{[0]} = 2\nabla_+ \tag{2.6}$$

$$\{J_{3+}, J_{3+}\}^{[0]} = -2\mathrm{ad}(J_{2+}) \,, \tag{2.7}$$

in the following sense: if $F(J_{0+}, J_{3+})$ and $G(J_{0+}, J_{3+})$ are two functionals on the light cone phase space, then their Poisson bracket is

$$\{F, G\} = \int d\tau^+ \, \mathrm{str} \left(2 \frac{\delta F}{\delta J_{0+}} \nabla_+ \frac{\delta G}{\delta J_{0+}} - 2 \frac{\delta F}{\delta J_{3+}} \left[J_{2+}, \frac{\delta G}{\delta J_{3+}} \right] \right), \tag{2.8}$$

with all the other components zero. In particular, the Poisson bracket of J_{2+} with everything else is zero. This means that this Poisson bracket is a degenerate one, and we have to restrict on the symplectic leaves, see the discussion in [7] for details. On a symplectic leaf we have

$$J_{2+}(\tau^+) = J_{2+}^{[0]}(\tau^+), \tag{2.9}$$

where $J_{2+}^{[0]}(\tau^+)$ is a fixed matrix-valued function. A convenient choice is:

$$J_{\bar{2}+} = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{\mathfrak{so}(2,4)} \bigoplus \begin{pmatrix} 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{\mathfrak{so}(6)} \tag{2.10}$$

where α_1 and α_2 are some constants. In string theory we want $J_{2+}^{[0]}$ to satisfy the Virasoro constraints

$$\mathrm{str} \, J_{2+}^2 = 0. \tag{2.11}$$

therefore we put:

$$\alpha_1 = \alpha_2 \quad (= \mathrm{const}).$$

Even after we fix J_{2+} as in (2.10), still $\theta^{[0]}$ is degenerate. To completely specify the symplectic leaf we fix in addition J_{3+} to be of the form:

$$J_{3+} - J_{3+}^{(0)} = [J_{2+}, K_1], \tag{2.12}$$

with fixed $J_{3+}^{(0)}$. In the theory of classical superstring in $AdS_5 \times S^5$ the symplectic leaves of the boost-invariant Poisson bracket are transversal to the orbits of the worldsheet reparametrizations and kappa-transformations. As explained in section 4.3 of [7] we can choose the kappa-gauge so that $J_{3+}^{(0)} = 0$, in other words

$$J_{3+} = [J_{2+}, K_1]. \tag{2.13}$$

On this symplectic leaf the symplectic form can be written as follows

$$\Omega^{[0]} = \int d\tau^+ \left(\mathrm{tr} \left(\delta f f^{-1} \delta (\partial_+ f f^{-1}) \right) + \mathrm{tr} \left(\delta K_{\bar{1}} [J_{\bar{2}+}, \delta K_{\bar{1}}] \right) \right). \tag{2.14}$$

where f is related to J_{0+} by the formula

$$J_{0+} = -\partial_+ f f^{-1}. \tag{2.15}$$

The discussion in [7] was limited to the positive component of the lightcone: $\tau^- = 0$. To completely describe the worldsheet, we have to specify a second characteristic, that is the negative component of the lightcone: $\tau^+ = 0$. We can choose

$$J_{1-} = [J_{2-}, K_3]\,, \tag{2.16}$$

on the negative component. Then the equations of motion are compatible with (2.13) and (2.16) in the following way

$$\nabla_- K_1 = J_{1-} + X_{1-} \tag{2.17}$$

$$\nabla_+ K_3 = J_{3+} + X_{3+}\,, \tag{2.18}$$

where X_{1-} and X_{3+} are undetermined quantities with the property $[J_{2+}, X_{1-}] = [J_{2-}, X_{3+}] = 0$.

3. The action giving rise to the boost-invariant Poisson bracket

In this section we rewrite the classical string equations of motion in a form which closely resembles the equations of motion of a gauged WZW model with a mass term. Then we will explain what is precisely the relation.

3.1 The action of Bakas, Park and Shin

3.1.1 An equivalent form of the string worldsheet equations of motion.

As a warmup let us consider the bosonic string on $\mathbb{R} \times S^n$. The sphere S^n is the symmetric space $\mathrm{SO}(n+1)/\mathrm{SO}(n)$. We denote:

$$G = \mathrm{SO}(n+1) \quad , \quad G_0 = \mathrm{SO}(n) \quad , \quad H = \mathrm{SO}(n-1)\,. \tag{3.1}$$

The corresponding Lie algebras are:

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 = \mathfrak{so}(n+1)$$
$$\mathfrak{g}_0 = \mathfrak{so}(n)$$
$$\mathfrak{h} = \mathfrak{so}(n-1)\,.$$

The equations of motion are

$$\nabla_+ J_{2-} = \nabla_- J_{2+} = 0 \tag{3.2}$$

$$[\nabla_+, \nabla_-] + [J_{2+}, J_{2-}] = 0\,, \tag{3.3}$$

where $\nabla_\pm = \partial_\pm + J_{0\pm}$. We can choose such a gauge that $J_{2+} = T$ is a constant matrix (c.f. eq. (2.10)). For example for $n = 5$ we can take:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the stabilizer of T in $\mathfrak{g}_0$ is $\mathfrak{h} = \mathfrak{so}(n-1)$. Then (3.2) implies that

$$J_{0-} = A_- \in \mathfrak{h}\,. \tag{3.4}$$

Let us introduce $g \in G_0$ such that

$$J_{2-} = g^{-1} J_{2+} g\,. \tag{3.5}$$

Then eq. (3.2) implies that

$$\partial_+ + J_{0+} = g^{-1}(\partial_+ + A_+)g\,,\ A_+ \in \mathfrak{h}\,. \tag{3.6}$$

Therefore the phase space of the classical string can be described by the data

$$(g\,,\ A_+\,,\ A_-)\,, \tag{3.7}$$

subject to the equations

$$[g^{-1}(\partial_+ + A_+)g\,,\ \partial_- + A_-] + [T, g^{-1}Tg] = 0\,, \tag{3.8}$$

modulo the gauge symmetries

$$
\begin{aligned}
g &\mapsto h_L g h_R^{-1} \\
\partial_+ + A_+ &\mapsto h_L(\partial_+ + A_+)h_L^{-1} \\
\partial_- + A_- &\mapsto h_R(\partial_- + A_-)h_R^{-1}\,.
\end{aligned} \tag{3.9}
$$

The h_L gauge symmetry is a "tautological" gauge symmetry, existing because we replaced J_{2-} with g, see eq. (3.5). And the h_R gauge symmetry is what remains of (2.2), after we put $J_{2+} = T$.

3.1.2 Gauged WZW with a mass term

Eq. (3.8) is identified in [4] as one of the equations of motion of the mass deformed gauged WZW model with the gauge fields A_+ and A_-. More precisely, the action takes the form

$$S_{BPS}(g, A_+, A_-) = S_{WZW}(g) + S_{\text{gauge}}(g, A_+, A_-) + S_{\text{mass}}(g)\,, \tag{3.10}$$

where

$$
\begin{aligned}
S_{WZW} &= -\frac{1}{4\pi}\left(\int d^2\tau \operatorname{Tr}(\partial_+ g \partial_- g^{-1}) + \int_B \frac{1}{3}\operatorname{Tr}(g^{-1}dg)^3\right) \\
S_{\text{gauge}} &= \frac{1}{2\pi}\int d^2\tau \operatorname{Tr}\left(A_+ A_- - A_+ g A_- g^{-1} + A_+ \partial_- g g^{-1} - A_- g^{-1}\partial_+ g)\right) \\
S_{\text{mass}} &= \frac{1}{2\pi}\int d^2\tau \operatorname{Tr}\left(T g^{-1}Tg\right)\,.
\end{aligned} \tag{3.11}
$$

The variation with respect to g of the action $S_{BPS}(g, A_+, A_-)$ is

$$\delta S_{BPS} = \int \operatorname{Tr}\left(([g^{-1}(\partial_+ + A_+)g\,,\ \partial_- + A_-] + [T, g^{-1}Tg])g^{-1}\delta g\right)\,. \tag{3.12}$$

This leads to the equation of motion which is identical to (3.8):

$$[g^{-1}(\partial_+ + A_+)g \, , \, \partial_- + A_-] + [T, g^{-1}Tg] = 0 \,, \tag{3.13}$$

The variation with respect to A_+ and A_- gives the equations of motion for the gauge fields

$$A_+ = \left(g^{-1}(\partial_+ + A_+)g\right)_{\mathfrak{h}} \tag{3.14}$$

$$A_- = \left(g(\partial_- + A_-)g^{-1}\right)_{\mathfrak{h}} \,. \tag{3.15}$$

We will explain the relation between the classical string described by the equations of motion (3.7)–(3.9) and the gauged WZW model. The main idea is to observe that the classical string can be identified with the Hamiltonian reduction of the WZW model with respect to the symmetries (A.7). The Hamiltonian reduction of the WZW model is closely related to the gauged WZW model, in fact it coincides with the gauged WZW model up to subtleties with zero modes.

3.2 Relation between string worldsheet theory and gauged WZW: formal analysis on an infinite worldsheet

Classical solutions of the action (3.10) are also solutions of eqs. (3.8). It is not immediately obvious why all the solutions of (3.7)–(3.9) can be obtained as classical solutions of (3.10), because there are additional equations (3.14) and (3.15). In other words, the action (3.10) gives solutions of the system (3.7)–(3.9) with the particular $A_\pm$, namely $A_\pm$ satisfying (3.14) and (3.15). One has to prove that any solution of (3.8) can be transformed by the gauge transformations (3.9) to a solution satisfying (3.14) and (3.15). The detailed analysis of this question in both bosonic and supersymmetric cases is discussed in [8]. Let us briefly summarize the argument from our point of view. We have to prove that all the solutions of the system (3.7)–(3.9) can be obtained from (3.10). Given an arbitrary solution of the system (3.7)–(3.9), we can bring it to the gauge $A_\pm = 0$ using the gauge transformations (3.9) with the parameters $h_L = P \exp \int d\tau^+ A_+$ and $h_R = P \exp \int d\tau^- A_-$. In this gauge g satisfies

$$\partial_-(g^{-1}\partial_+ g) = [T, g^{-1}Tg] \,. \tag{3.16}$$

Moreover, even after we fix the gauge $A_\pm = 0$ there are still residual gauge transformations with $h_L = h_L(\tau^-)$ and $h_R = h_R(\tau^+)$. Let us first assume that the worldsheet is infinite, then we can use these gauge transformations to further fix the gauge, so that

$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \tag{3.17}$$

$$(\partial_- gg^{-1})_{\mathfrak{h}} = 0 \,. \tag{3.18}$$

This is possible because (3.16) implies that

$$j_+ = (g^{-1}\partial_+ g)_{\mathfrak{h}} \ \text{ and } \ j_- = (\partial_- gg^{-1})_{\mathfrak{h}} \,, \tag{3.19}$$

are holomorphic and antiholomorphic currents

$$\partial_- j_+ = \partial_+ j_- = 0 \,. \tag{3.20}$$

This means that $h_R = P \exp\left(-\int j_+ d\tau^+\right)$ is holomorphic and $h_L = P \exp\left(-\int j_- d\tau^-\right)$ is antiholomorphic and therefore we can use the residual gauge transformation with these h_L and h_R to fix $j_+ = j_- = 0$ which is precisely (3.17), (3.18). Now $(g, A_\pm)$ satisfies (3.14) and (3.15). This proves that any solution of the classical string equations (3.7)–(3.9) can be gauge transformed to satisfy (3.14) and (3.15), and therefore is also a solution of the equations of motion of (3.10).

3.3 Interpretation as Hamiltonian reduction of WZW model

In the next section we will include fermions and explain how the classical superstring in $AdS_5 \times S^5$ is related to the gauged WZW model interacting with fermions. But before we discuss fermions we want to give a "geometrical" explanation of why eq. (3.16) implies the existance of the holomorphic and antiholomorphic currents, eq. (3.20). This will be useful for understanding the fermionic extension. After we include fermions the holomorphic and antiholomorphic currents become more complicated, but the geometrical interpretation of them as moment maps remains the same.

Notice that eq. (3.16) is the equation of motion of the mass deformed (ungauged) WZW model with the action $S_{WZW}(g) + S_{\mathrm{mass}}(g)$. The classical phase space of the WZW model (with or without the mass term) has a symmetry:

$$g(\tau^+, \tau^-) \mapsto h_L(\tau^-) g(\tau^+, \tau^-) h_R(\tau^+)^{-1} \,. \tag{3.21}$$

Here $h_L(\tau^-)$ and $h_R(\tau^+)$ are periodic H-valued functions, so the symmetry group is the product of two loop groups: $LH \times LH$. Now we have a Hamiltonian system (the classical WZW model) and a symmetry acting on its phase space ($LH \times LH$). One can verify that this symmetry preserves the symplectic form, and in fact the currents j_+ and j_- defined by eqs. (3.19) are precisely the densities of the moment map corresponding to this symmetry, and eq. (3.20) is the conservation of the moment map. (See the appendix for details.)

Setting $j_\pm$ to zero (eqs. (3.17) and (3.18)) corresponds to considering the *Hamiltonian reduction* of the WZW model by the symmetry $LH \times LH$. From this point of view the classical string described by eqs. (3.7)–(3.9) is naturally identified, at least at the level of equations of motion, with the Hamiltonian reduction of the WZW model. The Hamiltonian reduction of the WZW model is closely related to the gauged WZW model (3.10), in fact it is equivalent to the gauged WZW model on the infinite line. On the cylinder there is a mismatch of zero modes, see the appendix for details.

Similar arguments hold for the fermionic extension which we will now describe.

4. Including fermions

We will now show that we can include fermions to the mass-deformed gauged WZW model so that the classical equations of motion agree with those of the Metsaev-Tseytlin string in $AdS_5 \times S^5$.

4.1 Fermionic terms in the action

The symplectic leaf (2.12) is parametrized by K_1. This means that the action of the generalized sine-Gordon model should be described in terms of K_1 and K_3 rather than J_3 and J_1. Therefore we will take K_1 and K_3 as independent variables. Eq. (2.14) suggests to look for an action in the following form:

$$
S = S_{BPS} + \Delta S_{\mathrm{kin}} + \Delta S_{\mathrm{mass}} = S_{BPS} + \int d^2\tau \left\{ -\frac{1}{2}\mathrm{str}[J_{2+}, K_1]\nabla_- K_1 - \right.
$$
$$
\left. -\frac{1}{2}\mathrm{str}[J_{2-}, K_3]\nabla_+ K_3 + \mathrm{str}[J_{2+}, K_1][J_{2-}, K_3] \right\},
\tag{4.1}
$$

where S_{BPS} is described in the previous section. Let us consider the gauge where $J_{2+} = T$ and $J_{2-} = g^{-1}Tg$ and T is a constant matrix. In this gauge

$$
\nabla_- = \partial_- + \mathrm{ad}(J_{0-}) = \partial_- + \mathrm{ad}(A_-)
\tag{4.2}
$$
$$
\nabla_+ = \partial_+ + \mathrm{ad}(J_{0+}) = \partial_+ + \mathrm{ad}(g^{-1}A_+ g + g^{-1}\partial_+ g).
\tag{4.3}
$$

We will now prove that this action leads to the correct equations of motion for the classical superstring in $AdS_5 \times S^5$.

4.2 Equations of motion

We now derive the equations of motion from the variation of $K_{1,3}$ and g and show that these agree with the string equations of motion.

First consider the variation with respect to the fields K_1 and K_3, which will yield the equations of motion for the fermionic fields. Varying δK_1 gives

$$
-[J_{2+}, \nabla_- K_1] + [J_{2+}, [J_{2-}, K_3]] = -\nabla_- J_{3+} - [J_{1-}, J_{2+}] = 0.
\tag{4.4}
$$

Likewise the variation with respect to K_3 yields

$$
-[J_{2-}, \nabla_+ K_3] + [J_{2-}, [J_{2+}, K_1]] = -\nabla_+ J_{1-} - [J_{3+}, J_{2-}] = 0.
\tag{4.5}
$$

These are the correct equations of motion for the fermions.

The bosonic equations are obtained from the variation $\delta_\xi g = g\xi$. We have

$$
\delta_\xi \nabla_- = 0, \quad \delta_\xi \nabla_+ = \mathrm{ad}(\nabla_+ \xi), \quad \delta_\xi J_{2-} = [J_{2-}, \xi].
\tag{4.6}
$$

The ξ-variation of the BPS action gives

$$
\delta_\xi S_{\mathrm{BPS}} = \int \mathrm{str}\left(\xi\left(\partial_+ J_{0-} - \partial_- J_{0+} + [J_{0+}, J_{0-}] + [J_{2+}, J_{2-}]\right)\right).
\tag{4.7}
$$

The variation of ΔS_{kin} is

$$
\delta_\xi \Delta S_{\mathrm{kin}} = -\frac{1}{2}\int \mathrm{str}\,\xi \nabla_+[K_3, [K_3, J_{2-}]] + \frac{1}{2}\int \mathrm{str}[J_{2-}, [K_3, \nabla_+ K_3]] =
$$
$$
= -\int \mathrm{str}\,\xi[K_3, [\nabla_+ K_3, J_{2-}]] = \int \mathrm{str}\,\xi[K_3, [J_{2-}, J_{3+}]].
\tag{4.8}
$$

We used the fermion equation of motion (4.5), which implies that $\nabla_+ K_3 = J_{3+} +$ terms that are annihilated by $\mathrm{ad}(J_{2-})$. Finally, the variation of ΔS_{mass} is

$$\delta_\xi \Delta S_{\mathrm{mass}} = \int \delta_\xi \, \mathrm{str}[J_{2+}, K_1][J_{2-}, K_3] = -\int \mathrm{str}[[[J_{2+}, K_1], K_3], J_{2-}]\xi \,. \qquad (4.9)$$

We get:

$$\delta_\xi(\Delta S_{\mathrm{kin}} + \Delta S_{\mathrm{mass}}) = \mathrm{str}\,(\xi[J_{3+}, J_{1-}]) \,. \qquad (4.10)$$

Therefore the variation of $S_{BPS} + \Delta S_{\mathrm{kin}} + \Delta S_{\mathrm{mass}}$ gives the correct bosonic equation of motion

$$\partial_+ J_{0-} - \partial_- J_{0+} + [J_{0+}, J_{0-}] + [J_{2+}, J_{2-}] + [J_{3+}, J_{1-}] = 0 \,. \qquad (4.11)$$

We have shown that the action of (4.1) reproduces correctly all the equations of motion, (4.5), (4.4) and (4.11), of the GSMT super-string on $AdS_5 \times S^5$.

4.3 Variation with respect to $A_\pm$

The story is similar to the case of the bosonic string. As in section 3.1.2 we can go to the gauge where $A_\pm = 0$. The equations following from (4.1) are the same as the string equations of motion plus the vanishing of the holomorphic current

$$j_+ = \left(g^{-1}\partial_+ g - \frac{1}{2}[K_1, [K_1, J_{2+}]] \right)_{\mathfrak{h}} \,, \qquad (4.12)$$

and vanishing of the similar antiholomorphic current j_-. As in the case of the bosonic string j_+ and j_- can be gauged away by the residual holomorphic and antiholomorphic gauge transformations. The holomorphicity follows from the equations of motion, see [8] for a detailed discussion of these questions. From the point of view of the boost-invariant symplectic structure (2.14) the equation $j_+ = 0$ can be interpreted as skew-orthogonality to the orbits of the gauge transformations $\delta g = g\xi$, $\delta K_1 = [K_1, \xi]$, $\xi \in \mathfrak{h}$. Therefore j_+ can be understood as the moment map of the ungauged mass-deformed WZW with fermions with respect to the symmetry:

$$g(\tau^+, \tau^-) \mapsto g(\tau^+, \tau^-)h_R(\tau^+) \qquad (4.13)$$
$$K_{1,3}(\tau^+, \tau^-) \mapsto h_R(\tau^+)^{-1}K_{1,3}(\tau^+, \tau^-)h_R(\tau^+) \,. \qquad (4.14)$$

Acknowledgments

A. Hamiltonian reduction of WZW model

A.1 Classical string with periodic boundary conditions and Hamiltonian reduction of WZW

We have seen that on an infinite worldsheet the classical string is equivalent to the gauged WZW model. The analysis on an infinite worldsheet is formal because we neglect the boundary terms. What happens when we consider instead a cylindrical worldsheet? To understand periodic boundary conditions, we will use an interpretation of the classical string as a Hamiltonian reduction. It turns out that the classical string in the form described by eqs. (3.7)–(3.9) is closely related to the Hamiltonian reduction of the mass deformed WZW model with respect to the symmetries (A.7).

The precise relation is the following. Let $\mathcal{M}_{\text{string}}$ denote the space of classical solutions of the equations (3.7)–(3.9). It can be represented as a continuous family of subspaces $\mathcal{M}_{\text{string}}^{[m_L],[m_R]}$ parametrized by the conjugacy classes of the monodromies $m_L = \overrightarrow{P} \exp\left[-\int_0^{2\pi} \partial_- g g^{-1}|_{\mathfrak{h}} \, d\tau^-\right]$ and $m_R = \overrightarrow{P} \exp\left[\int_0^{2\pi} g^{-1}\partial_+ g|_{\mathfrak{h}} \, d\tau^+\right]$:

$$\mathcal{M}_{\text{string}} = \bigcup_{[m_L],[m_R]} \mathcal{M}_{\text{string}}^{[m_L],[m_R]}. \tag{A.1}$$

Each subspace $\mathcal{M}_{\text{string}}^{[m_L],[m_R]}$ is naturally identified as the phase space of a Hamiltonian reduction of the mass deformed WZW model:

$$\mathcal{M}_{\text{string}}^{[m_L],[m_R]} = \mathcal{M}_{WZW//(LH \times LH)}^{[m_L],[m_R]} \tag{A.2}$$

where $[m_L]$ and $[m_R]$ are identified the conjugacy classes of the moment map.

We will explain in section A.4 that the Hamiltonian reduction of the WZW model is closely related to the gauged WZW model.

A.2 Hamiltonian reduction of WZW model

Consider a classical mechanical system with the action of some group H on the phase space $\mathcal{M}$. Let $\mathfrak{h} = Lie(H)$ denote the Lie algebra of H. Suppose that H preserves the symplectic structure, and therefore it is generated by a set of Hamiltonians; each vector field $\xi \in \mathfrak{h}$ is generated by its own corresponding Hamiltonian $\mathcal{H}_\xi$. Notice that we should have

$$\{\mathcal{H}_{\xi_1}, \mathcal{H}_{\xi_2}\} = \mathcal{H}_{[\xi_1,\xi_2]} + \text{const}. \tag{A.3}$$

We think of the constant term as H_c where c is the central element of some central extension of $\mathfrak{h}$, let us call it $\hat{\mathfrak{h}}$. The *moment map* μ is a map from the phase space $\mathcal{M}$ to $(\hat{\mathfrak{h}})^*$, which is defined in the following way. For each point $x \in \mathcal{M}$, and $\xi \in \mathfrak{h}$, we define:

$$\mu(x) \in \hat{\mathfrak{h}}^* \;:\quad \langle \mu(x), \xi \rangle = H_\xi(x) \tag{A.4}$$

It follows from (A.3) that the moment map has the property of *equivariance*:

$$\mu(h.x) = \text{Ad}(h^{-1})^*.\mu(x). \tag{A.5}$$

The *Hamiltonian reduction* consists of three steps. First choosing a coadjoint orbit $\mathcal{O} \subset \mathfrak{h}^*$, then restricting to the subspace of the phase space determined by the equation $\mu(x) \in \mathcal{O}$, and finally identifying the points which are connected by the action of H:

$$x \simeq y \ \text{if} \ y = h.x \ \text{for some} \ h \in H \,.$$

Schematically:

$$\mathcal{M}//H = \mu^{-1}\mathcal{O}/H \,. \tag{A.6}$$

Notice that the Hamiltonian reduction depends on the choice of a coadjoint orbit $\mathcal{O} \subset \mathfrak{h}^*$.

Let us now look at the Hamiltonian reduction of the WZW model by the symmetries:

$$g(\tau^+, \tau^-) \mapsto h_L(\tau^-)g(\tau^+, \tau^-)h_R(\tau^+)^{-1} \tag{A.7}$$

where both h_L and h_R are in $H \subset G$. The symplectic structure of the (ungauged) WZW model is given by this equation:

$$\Omega_{WZW} = \int_0^{2\pi} d\tau^+ \ \text{tr} \ \delta g g^{-1}\partial_+(\delta g g^{-1}) \ - \ \int_0^{2\pi} d\tau^- \ \text{tr} \ g^{-1}\delta g\partial_-(g^{-1}\delta g) \,. \tag{A.8}$$

Notice that these symmetries form two copies of the loop group[1] of H; therefore the symmetry group is $LH \times LH$. The infinitesimal version of (A.7) is

$$\delta_{(\alpha_L, \alpha_R)}g(\tau^+, \tau^-) = \alpha_L(\tau^-)g - g\alpha_R(\tau^+) \,, \tag{A.9}$$

where the Lie algebra $L\mathfrak{h} \oplus L\mathfrak{h}$ is parametrized by the pair (α_L, α_R). The moment map is:

$$\langle \mu \,, \ (\alpha_L, \alpha_R) \rangle = -2 \int_0^{2\pi} d\tau^- \text{tr} \ \alpha_L\partial_-gg^{-1} - 2 \int_0^{2\pi} d\tau^+ \text{tr} \ \alpha_R g^{-1}\partial_+g \,. \tag{A.10}$$

This, in particular, implies that $\partial_+(\partial_-gg^{-1}|_{\mathfrak{h}}) = \partial_-(g^{-1}\partial_+g|_{\mathfrak{h}}) = 0$. We will denote:

$$j_+ = g^{-1}\partial_+g|_{\mathfrak{h}} \qquad \text{and} \qquad j_- = \partial_-gg^{-1}|_{\mathfrak{h}} \tag{A.11}$$

The coadjoint action of $\widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{h}}$ on j_+ and j_- is given by the formulas:

$$\begin{array}{ll} \delta_{\alpha_R}j_+ = -\partial_+\alpha_R - [j_+, \alpha_R] & \delta_{\alpha_R}j_- = 0 \\ \delta_{\alpha_L}j_+ = 0 & \delta_{\alpha_L}j_- = -\partial_-\alpha_L - [j_-, \alpha_L] \,. \end{array} \tag{A.12}$$

Since we want to discuss the Hamiltonian reduction of the WZW model, we need to know what are the orbits of this coadjoint action. To describe the orbits we need to classify the invariants of this action. The invariants are the eigenvalues of the left and right monodromy matrices. These monodromy matrices are defined as follows:

$$m_L = \overrightarrow{P} \exp\left[-\int_0^{2\pi} j_-d\tau^-\right] \tag{A.13}$$

$$m_R = \overrightarrow{P} \exp\left[\int_0^{2\pi} j_+d\tau^+\right] \,. \tag{A.14}$$

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[1]The elements of the loop group LH are group-valued functions $h(\sigma)$ satisfying $h(\sigma + 2\pi) = h(\sigma)$

A.3 How the space of solutions to (3.7)–(3.9) is related to the Hamiltonian reduction of WZW by the symmetries (A.7)?

Let us look at the solutions to the system of equations (3.7)–(3.9). We denote this space $\mathcal{M}_{\text{string}}$. Just as we did on the infinite line, we can still gauge away A_+ and A_- on the cylinder using the gauge transformations (3.9); there is no obstacle. In the gauge $A_\pm = 0$ the equation (3.8) becomes the WZW equation of motion and the residual gauge transformations are precisely the symmetries (A.7) which we used to define the Hamiltonian reduction. On an infinite line we could use these residual gauge transformations to put $j_\pm = 0$, but on a cylinder the conjugacy classes of m_L and m_R (defined in eqs. (A.13) and (A.14)) are obstacles to gauging away $j_\pm$. The space of solutions splits into a union of subspaces with a fixed conjugacy classes of m_L and m_R:

$$\mathcal{M}_{\text{string}} = \bigcup_{([m_L],[m_R]) \in H/H \times H/H} \mathcal{M}_{\text{string}}^{[m_L],[m_R]}. \tag{A.15}$$

For every fixed $[m_L]$ and $[m_R]$ the subspace $\mathcal{M}_{\text{string}}^{[m_L],[m_R]}$ is identical to the phase space of the Hamiltonian reduction of the WZW model on the value of the moment map corresponding to $([m_L], [m_R])$.

A.4 How the Hamiltonian reduction of WZW is related to the gauged WZW?

We want to explain in which sense the action of the massive gauged WZW given by eq. (3.10) describes the classical string. We already explained how the classical string is related to the Hamiltonian reduction of the WZW model by the infinite dimensional symmetry group $LH \times LH$ acting as specified in (A.7). But what is the relation between the Hamiltonian reduction of the WZW model and the gauged WZW model? It turns out that these two models are equivalent modulo subtleties with zero modes, which we will now describe.

We want to understand the relation between these two systems:

1. Gauged WZW model defined by the action $S_{WZW} + S_{\text{gauge}}$ (see eq. (3.11))

2. Hamiltonian reduction of (ungauged) S_{WZW} with respect to the symmetries:

$$g(\tau^+, \tau^-) \mapsto h_L(\tau^-) g(\tau^+, \tau^-) h_R(\tau^+)^{-1} \tag{A.16}$$

As we explained, the procedure of Hamiltonian reduction depends on the choice of a conjugacy class of m_L and the choice of a conjugacy class of m_R. In particular if $m_L = m_R = \mathbf{1}$, then we can use the gauge transformations (A.12) to choose g to satisfy (3.17) and (3.18). From this point of view eq. (3.17) defines a submanifold in the phase space which is skew-orthogonal with respect to the Kirillov form (A.8) to the orbit of the gauge transformations $g \mapsto g h_R^{-1}$, see section 5.6 of [7]. Similarly (3.18) defines a subspace orthogonal to the orbit of $g \mapsto h_L g$.

More generally, suppose that the conjugacy class of j_+ under the transformation δ_{α_R} coincides with the conjugacy class of j_- under the transformation δ_{α_L}. This means that there is $f \in H$ such that

$$f m_L f^{-1} = m_R. \tag{A.17}$$

In this case we will denote

$$\mathcal{M}_{\text{string}}^{[m]} = \mathcal{M}_{\text{string}}^{[m],[m]}\,.$$

It turns out that $\mathcal{M}_{\text{string}}^{[m]}$ can be identified as the Hamiltonian reduction of the gauged WZW phase space on the level set of the conjugacy class of the holonomy of the WZW gauge field A (the A of eq. (3.11)).

Indeed, let us describe the map from g of (A.8) to a solution of the gauged WZW model. First of all, making the constant gauge transformation with $f \in H$ we can choose

$$m_L = m_R = m\,, \tag{A.18}$$

and we can also rotate m into a fixed maximal torus of H. Then consider $\widehat{g}$ defined by the formula:

$$\widehat{g}(\tau,\sigma) = \left(\overrightarrow{P} \exp\left[-\int_0^\sigma j_- d\tau^- \right] \right) g \left(\overleftarrow{P} \exp\left[-\int_0^\sigma j_+ d\tau^+ \right] \right)\,. \tag{A.19}$$

Notice that $\widehat{g}$ has the following properties:

$$\partial_-(\widehat{g}^{-1}\partial_+\widehat{g}) = 0 \tag{A.20}$$

$$\widehat{g}^{-1}\partial_+\widehat{g}|_{\mathfrak{h}} = \partial_-\widetilde{g}\widetilde{g}^{-1}|_{\mathfrak{h}} = 0 \tag{A.21}$$

$$\widehat{g}(2\pi) = m\widehat{g}(0)m^{-1}\,. \tag{A.22}$$

This is almost what we need, except for we want to make g periodic. Consider $\mu \in \mathfrak{h}$ such that $m = e^{2\pi\mu}$, and define $\tilde{g}$:

$$\tilde{g}(\tau,\sigma) = e^{-\sigma\mu}\widehat{g}(\tau,\sigma)e^{\sigma\mu} \tag{A.23}$$

One can check that $\tilde{g}$ satisfies the equations of motion (3.13), (3.14) and (3.15) of the gauged WZW model with $A_\tau = 0$ and $A_\sigma = \mu$.
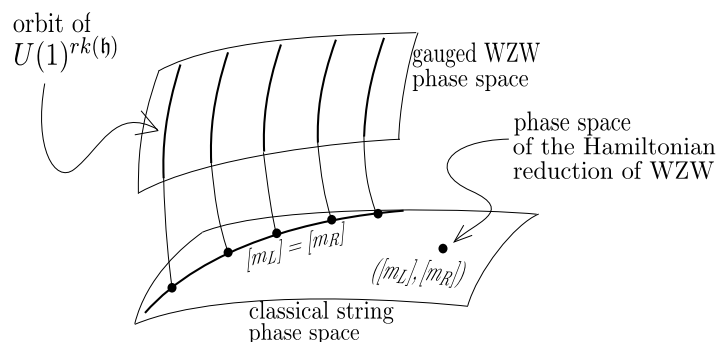
On the other hand, notice that any solution of the gauged WZW can be gauged to $A_\tau = 0$, $A_\sigma = \mu$ for some μ. The conjugacy class $[\mu]$ is a dynamical variable in the gauged WZW. But when we do the Hamiltonian reduction of the WZW model, we fix $[\mu]$. Moreover, the Hamiltonian reduction of the WZW model has additional residual gauge tranformations which correspond to the following transformations of $\tilde{g}$:

$$\delta\tilde{g} = \alpha\tilde{g} + \tilde{g}\alpha \quad , \qquad \alpha \in \mathfrak{h} \tag{A.24}$$

where the gauge parameter α should commute with μ: $[\alpha,\mu] = 0$. From the point of view of the gauged WZW model these transformations are generated by the eigenvalues of $P \exp \int A$, i.e. the eigenvalues of μ.

Indeed, the symplectic form following from the action $S_{WZW}(g) + S_{\text{gauge}}(g, A_+, A_-)$ is:

$$\Omega = \int d\tau^+ \, \text{tr} \, \left(\delta gg^{-1}\nabla_+(\delta gg^{-1}) + 2\delta gg^{-1}\delta A_+ \right) +$$
$$+ \int d\tau^- \, \text{tr} \, \left(-g^{-1}\delta g\nabla_-(g^{-1}\delta g) + 2g^{-1}\delta g\delta A_- \right)\,. \tag{A.25}$$

Figure 1: The classical string phase space is shown as a horizontal plane; each point on the plane corresponds to the phase space of the Hamiltonian reduction of WZW for the value of the moment map $([m_L], [m_R])$. The phase space of the gauged WZW is mapped on the subspace $[m_L] = [m_R]$ of codimension $rk(\mathfrak{h})$. The map involves identification of the points related by the action of $\mathrm{U}(1)^{rk(\mathfrak{h})}$.

We can choose the gauge where $A_+ = -A_- = \mu$ and μ belongs to the Cartan subalgebra of $\mathfrak{h}$. In this gauge it is straightforward to see that the Hamiltonian $\mathrm{tr}(\alpha\mu)$ generates (A.24).

We demonstrated that the Hamiltonian reduction of the WZW model on the fixed value μ of the moment map corresponds to the gauged WZW with the fixed $P \exp \int A = e^{2\pi\mu}$ with the following identification: two configurations are considered equivalent when they are related by the transformation (A.24). But this is precisely the Hamiltonian reduction of the gauged WZW model on a fixed value of the conjugacy class of the holonomy $P \exp \int A$.

The conclusion is that the Hamiltonian reduction of the WZW model by the infinite-dimensional group $LH \times LH$ acting according to eq. (A.16) is equivalent to the Hamiltonian reduction of the gauged WZW model by the finite-dimensional group $\mathrm{U}(1)^{rk(\mathfrak{h})}$ generated by the eigenvalues of the holonomy of the gauge field.

Figure 1 illustrates the relation between the phase spaces.

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